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Superordination of Analytic Functions of Koebe Type

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Abstract We investigate the new class $H_b(\alpha,\beta_i)$ of analytic functions with Koebe type. The subordination, superordination, best dominant result and the sandwich theorem for that class are obtained.

Keywords: Sandwich-type theorem, superordination, subordination, analytic functions, Koebe type.

1. Introduction

Let denote by $H(U)$ the space of all analytic functions in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. If $f,F,\in H(U)$ and F is univalent in U we say that the function f is subordinated to F, or F is superordinated to f, written $f(z)\prec F(z)$, if $f(0) = F(0)$ and $f(U)\subseteq F(U)$. For $a\in\mathbb{C}$ and $n\in\mathbb{N}^*$ we denote $H[a,\ n]=\{f\in H(U): f(z)=a+a_n z^n+\ldots\}$.

Let $\phi:\mathbb{C}^3\times U\to\mathbb{C}$, $h\in H(U)$ and $q\in H[a,\ n]$, in Miller and Mocanu (2003), the authors determined conditions on ϕ such that $h(z)\prec\phi\left(p(z), zp'(z), z^2p''(z); z\right)$ implies $q(z)\prec p(z)$, for all p functions that satisfy the above superordination. Moreover, by founding sufficient conditions so that the q function is the largest function with this property, called the best subordinant of this superordination. Let A denote the class of normalized analytic functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1}$$

which are analytic in the open unit disk $U=\{z\ :|z|<1\}$. Also as usual, let

$$S^*=\left\{f: f\in A,\ \left(\frac{zf'(z)}{f(z)}\right)>0,\ (z\in U)\right\} \tag{2}$$

and

$$S_{st}^{\ *}=\left\{f: f\in A,\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2},\ (z\in U),\right\} \tag{3}$$

$(0<\alpha\le1)$, be the familiar classes of starlike functions in U and strongly starlike functions of order α in U, respectively. In this paper we consider and denote by $H(\alpha,\beta_i)$ the class of functions $f\in A$, for $\alpha>0, |\beta_i|<1$, that is

$$\operatorname{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{\beta+\gamma}\left(\frac{zf'(z)}{f(z)}+\alpha z^2\frac{zf''(z)}{f'(z)}\right)\right\}>0,\ z\in U. \tag{4}$$

We have the following inclusion relationships:
(i) $H(0,0) \subset S^*(0)$.
(ii) $H(1,0) \subset ST(\frac{1}{2})$, have investigated by Ramesha et al. (1995).
(iii) $H(1,0) \subset ST(\gamma)$, where $(\gamma < \frac{1}{2})$ already observed by Nunokawa et al. (1996).
(iv) $H(\alpha,0) \subset S^*$, already discussed by Kamali and Srivastava (2004).
(v) $H(\alpha,\beta) \subset S^*$, have investigate by Saibah Siregar (2011).

We consider the function f_b of Koebe type, defined by

$$f_b(z) := \frac{z}{(1-z^n)^b} \left(b \ge 0; n \in \mathbb{N} := \{1,2,3,\dots\}\right) \tag{5}$$

which obviously corresponds to the familiar Koebe function when $n=1$ and $b=2$. From (4) and (5) we denote by $H_b(\alpha,\beta_i)$, the class of A defined by

$$\operatorname{Re}\left\{\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\gamma}\left(\frac{zf_b'(z)}{f_b(z)} + \alpha z^2 \frac{zf_b''(z)}{f_b'(z)}\right)\right\} > 0, \quad z \in U. \tag{6}$$

In this paper, we investigate the subordination and superordination, best dominant, best subordinant, sandwich theorem of the class the by $H_b(\alpha,\beta_i)$.
The motivation of subordination and superordination, best dominant, best subordinant, sandwich theorem of this paper are by Saibah Siregar et al. (2010), by Bansal and Raina (2010).

2. Preliminaries

Lemma 2.1 Let $q(z)$ be univalent in U and let the function θ and ϕ be analytic in a domain U containing $q(U)$, with $q(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = \gamma zq'(z)\,\phi(q(z))$, $\gamma > 0$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(i) $Q(z)$ is univalent and starlike in U
(ii) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0,\ (z \in U)$.

If $p(z)$ is analytic in U with $p(0) = q(0) = 1$, $p(U) \subset D$ and $\theta(p(z)) + \gamma zp'(z)\phi(p(z)) \prec \theta(q(z)) + \gamma zq'(z)\phi(q(z)) = h(z)$ then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.2 Let $q(z)$ be univalent in U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that

(i) $zq'(z)\varphi(q(z))$ is univalent and starlike in U

(ii) $\operatorname{Re}\left(\frac{\vartheta'(q(z))}{\varphi(q(z))}\right) > 0,\ (z \in U).$

If $p(z) \in H[q(0),1] \cap Q$ with $p(U) \subseteq D$ and $\vartheta p(z) + zp'(z)\varphi(p(z))$ is univalent in U and $\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$ then $q(z) \prec p(z)$ and q is the best subordinant.

3. Subordination and Superordination Results

Theorem 3.1 *Let* $f(z) \in A$ *satisfy* $f(z) \neq 0, (z \in U)$. *Also let the function* $q(z)$ *be univalent in* U *with* $q(0) = 1$ *and* $q(z) \neq 0$, *such that*

$$\operatorname{Re}\left(1 + (\beta+\gamma-1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0,\ (z \in U) \tag{7}$$

and

$$\operatorname{Re}\Big\{[1+(1-\alpha)(\beta+\gamma)] + (\beta+\gamma+1)(q(z) + \\ + (\beta+\gamma-1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)})\Big\} > 0,\ \ (z \in U) \tag{8}$$

For $|\beta+\gamma| \leq 1$ *and* $\alpha > 0$. *If*

$$\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\gamma}\left(\frac{zf_b'(z)}{f_b(z)} + \alpha z^2\frac{zf_b''(z)}{f_b'(z)}\right) \prec h(z),\ (z \in U), \tag{9}$$

where

$$h(z) = \alpha[q(z)]^{\beta+\gamma+2} + (1-\alpha)[q(z)]^{\beta+\gamma+1} + \alpha zq'(z)[q(z)]^{\beta+\gamma} \tag{10}$$

then $\left(\frac{zf_b'(z)}{f_b(z)}\right) \prec q(z),\ (z \in U)$, *and* $q(z)$ *is the best dominant of (9).*

Proof Suppose

$$p(z) = \frac{zf'_b(z)}{f_b(z)},\qquad \theta(w) = w^{\beta+\gamma}((1-\alpha)w + \alpha w^2) \\ \text{and}\quad \phi(w) = w^{\beta+\gamma} \tag{11}$$

then $\theta(w)$ and $\phi(w)$ exist analytic inside the domain $D^* = \mathbb{C}\setminus\{0\}$ which contains $q(U)$, $q(0) = 1$ and $\phi(w) \neq 0$ when $w \in q(U)$. Define the functions $Q(z)$ and $h(z)$ by

$$Q(z) = \alpha\, zq'(z)\ \phi((q(z)) = \alpha\, zq'(z)[q(z)]^{\beta+\gamma}$$

and

$$h(z)=\theta(q(z))+Q(z)=\alpha[q(z)]^{\beta+\gamma+2}+(1-\alpha)[q(z)]^{\beta+\gamma+1}+\alpha zq'(z)[q(z)]^{\beta+\gamma} \qquad (12)$$

then it follow from (7) and (8) that $Q(z)$ is starlike in U and $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)>0$, $(z\in U)$.

We note that the function $p(z)$ is analytic in U, with $p(0)=q(0)=1$. Since $0\notin p(U)$, therefore $p(U)\subset D^*, \alpha>0$ and hence, the hypothesis of Lemma 2.1 are satisfied. Applying Lemma 2.1, we find that

$$\begin{aligned}
&\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\gamma}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{zf_b''(z)}{f_b'(z)}\right)\\
&=\alpha\left[p(z)\right]^{\beta+\gamma+2}+(1-\alpha)\left[p(z)\right]^{\beta+\gamma+1}+\alpha zp'(z)\left[p(z)\right]^{\beta+\gamma}\\
&=\theta\left(p(z)\right)+\alpha zp'(z)[p(z)]^{\beta+\gamma}\\
&\prec h(z)=\alpha\left[q(z)\right]^{\beta+\gamma+2}+(1-\alpha)\left[q(z)\right]^{\beta+\gamma+1}+\alpha zq'(z)\left[p(z)\right]^{\beta+\gamma}\\
&=\theta\left(q(z)\right)+\alpha zq'(z)[q(z)]^{\beta+\gamma}, \quad (z\in U)
\end{aligned}$$

which implies that $\left(\frac{zf_b'(z)}{f_b(z)}\right)\prec q(z)$, $(z\in U)$ and prove that is the best dominant of (9).

□

For $\gamma=0$, the next remark will be obtained.

Remark 3.2 (Saibah Siregar, (2011)) *Let* $f(z)\in A$ *satisfy* $f(z)\neq 0$, $(z\in U)$. *Also let the function* $q(z)$ *be univalent in* U *with* $q(0)=1$ *and* $q(z)\neq 0$, *such that*

$$\operatorname{Re}\left(1+(\beta-1)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right)>0, \ (z\in U) \qquad (13)$$

and

$$\operatorname{Re}\left\{[1+(1-\alpha)\beta]+(\beta+1)\left(q(z)+(\beta-1)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right)\right\}>0, \qquad (z\in U). \qquad (8)$$

For $|\beta|\le 1$ *and* $\alpha>0$. *If*

$$\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{zf_b''(z)}{f_b'(z)}\right)\prec h(z), \ (z\in U), \qquad (14)$$

where

$$h(z)=\alpha[q(z)]^{\beta+2}+(1-\alpha)[q(z)]^{\beta+1}+\alpha zq'(z)[q(z)]^{\beta} \tag{15}$$

then $\left(\frac{zf_b'(z)}{f_b(z)}\right)\prec q(z),\ (z\in U)$, *and* $q(z)$ *is the best dominant of* (14). □

Theorem 3.3 *Let* f *be analytic in* U *such that* $f(0)=0$, h *be convex univalent in* U *and* $h\in H[0,1]\cap Q$.

Assume that

$$\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\gamma}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{zf_b''(z)}{f_b'(z)}\right)$$

is a univalent function in U, *where* $|\beta+\gamma|\le 1$ *and* $\alpha>0$.

If $h\in A$ *and the subordination*

$$\begin{aligned} h(z) &= \theta(q(z))+\alpha zq'(z)[q(z)]^{\beta+\gamma} \\ &\prec \left(\frac{zf'_b(z)}{f_b(z)}\right)^{\beta+\gamma}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{zf_b''(z)}{f_b'(z)}\right), \end{aligned}$$

holds then $q(z)\prec\left(\frac{zf_b'(z)}{f_b(z)}\right)$ *implies that* $q(z)\prec p(z)$ *where* $p(z)=\left(\frac{zf_b'(z)}{f_b(z)}\right)$ *and* q *is the best subordinant.*

Proof. Our aim is to apply Lemma 2.2. Setting

$$p(z)=\frac{zf'_b(z)}{f_b(z)},\qquad \theta(w)=w^{\beta+\gamma}((1-\alpha)w+\alpha w^2)$$

$$\text{and}\quad \phi(w)=w^{\beta+\gamma} \tag{16}$$

then $\theta(w)$ and $\phi(w)$ are analytic inside the domain $D^*=\mathbb{C}\setminus\{0\}$ which contains $p(0)=q(0)=1$ and $\varphi(w)\neq 0$ when $w\in p(U)$. Observed that $\theta(w)$, $\phi(w)$, are analytic in $\mathbb{C}$. Thus

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}>0.$$

Now, we must show that

$$\begin{aligned} h(z) &= \theta(q(z))+\alpha zq'(z)\phi(q(z)) \\ &\prec \theta(p(z))+\alpha zp'(z)\phi(p(z)). \end{aligned}$$

By the assumption of the theorem

$$h(z)=\theta(q(z))+\alpha zq'(z)\phi[q(z)]^{\beta+\gamma}$$

$$=\alpha\left[q(z)\right]^{\beta+\gamma+2}+(1-\alpha)\left[q(z)\right]^{\beta+\gamma+1}+\alpha zq'(z)\left[q(z)\right]^{\beta+\gamma}$$
$$\prec\alpha\left[p(z)\right]^{\beta+\gamma+2}+(1-\alpha)\left[p(z)\right]^{\beta+\gamma+1}+\alpha zp'(z)\left[p(z)\right]^{\beta+\gamma}=\theta\left(p(z)\right)+\alpha zp'(z)\phi\left[p(z)\right]^{\beta+\gamma}$$
$$=\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\gamma}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{zf_b''(z)}{f_b'(z)}\right).$$

Thus $q(z)\prec p(z)$ so that will implies $q(z)\prec\left(\frac{zf_b'(z)}{f_b(z)}\right)$ and q is the best subordinant. □

For $\gamma=1$, , the next remark will be obtained.

Remark 3.4 (Saibah Siregar, (2011)) *Let f be analytic in* U *such that* $f(0)=0$, h *be convex univalent in* U *and* $h\in H[0,1]\cap Q$.
Assume that

$$\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{zf_b''(z)}{f_b'(z)}\right)$$

is a univalent function in U, *where* $|\beta|\leq 1$ *and* $\alpha>0$.
If $h\in A$ *and the subordination*

$$h(z)=\theta(q(z))+\alpha zq'(z)[q(z)]^{\beta}$$
$$\prec\left(\frac{zf'_b(z)}{f_b(z)}\right)^{\beta}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{zf_b''(z)}{f_b'(z)}\right),$$

holds then $q(z)\prec\left(\frac{zf_b'(z)}{f_b(z)}\right)$ *implies that* $q(z)\prec p(z)$ *where* $p(z)=\left(\frac{zf_b'(z)}{f_b(z)}\right)$ *and* q *is the best subordinant.*

If we combination of Theorem 3.1 with Theorem 3.3 then we obtain the differential ***Sandwich-Type Theorem.*** □

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