Selangor Science &Technology Review
Special Issue: Transformation of Manufacturing in Digital Age
Vol. 5, No. 1, (2021)

Error Review of Single Iteration Explicit Approximations of Colebrook's Equation

Nurudin A.S [1*]

[1]Department of Engineering, Faculty of Engineering and Life Sciences, Universiti Selangor, Timur Tambahan, 45600 Batang Berjuntai, Selangor
nurudin@unisel.edu.my

Abstract: This paper reviews the common single iteration of explicit equations for estimating the friction factor in pipes. The friction factor values were computed using Microsoft Excel. Using absolute error, relative percentage error, mean absolute error (MAE), mean square error (MSE), and root mean square error (RMSE), the Colebrook's equation comparison was expressed. The best equation to estimate the friction factor was Beluco-Schettini when looking at the average error, MSE, and RMSE. In contrast, of all the equations, the Haaland equation is the most consistent.

Keywords: Colebrook's equation, Darcy friction factor

1. Introduction

Colebrook's equation (1), also known as the Colebrook-White equation, developed in 1939 to calculate the friction factor for flow in pipes problem (Colebrook, 1939). The development of the equation was considered a good achievement at the time. This equation's main disadvantage is because of its implicitness, making the equation considered challenging to solve. It requires the advanced mathematic level to solve the equation.

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right) \qquad (1)$$

From the 1930s until now, researchers developed several equations as an alternative for Colebrook's equation. In this paper, only explicit with a single iteration is reviewed. Table 1 shows the equations used in this paper, together with the range of validity of the equation.

Table 1. Various single iteration explicit approximations of Colebrook's equation

Equation no.	Equation Validity range	Author [reference]
(2)	$f = \left[-1.8\log\left(\varepsilon/D + \frac{7}{\mathrm{Re}}\right)\right]^{-2}$ Not specified	Altshul (Nekrasov, 1969)
(3)	$f = \left[-2\log\left(\frac{\varepsilon/D}{3.7} + \left(\frac{7}{\mathrm{Re}}\right)^{0.9}\right)\right]^{-2}$ Not specified	Churchill (Churchill, 1973)
(4)	$f = \left[-2\log\left(\frac{\varepsilon/D}{3.715} + \frac{15}{\mathrm{Re}}\right)\right]^{-2}$ Not specified	Eck (Winning & Coole, 2013)

Equation no.	Equation Validity range	Author [reference]
(5)	$f=\left[-2\log\left(\frac{\varepsilon/D}{3.715}+\left(\frac{6.943}{Re}\right)^{0.9}\right)\right]^{-2}$ 4 x $10^{-5} \le \varepsilon/D \le 5$ x 10^{-2} and 5 x $10^{3} \le Re \le 10^{7}$	Jain (Romeo et al., 2002)
(6)	$f=\left[-2\log\left(\frac{\varepsilon/D}{3.7}+\frac{5.74}{Re^{0.9}}\right)\right]^{-2}$ $10^{-6} \le \varepsilon/D \le 10^{-2}$ and 5 x $10^{3} \le Re \le 10^{8}$	Swamee - Jain (Swamee & Jain, 1976)
(7)	$f=\left[-1.8\log\left(0.135\,\varepsilon/D+\frac{6.5}{Re}\right)\right]^{-2}$ $0 \le \varepsilon/D \le 5$ x10^{-2} and 4 x $10^{3} \le Re \le 10^{8}$	Round (Round, 1980)
(8)	$f=\left[-2\log\left(\frac{\varepsilon/D}{3.7}+\left(\frac{6.81}{Re}\right)^{0.9}\right)\right]^{-2}$ Not specified	Pavlov – Romankov -Noskov (Levenspiel, 1998)
(9)	$f=\left[-1.8\log\left(\left(\frac{\varepsilon/D}{3.7}\right)^{1.11}+\frac{6.9}{Re}\right)\right]^{-2}$ $10^{-6} \le \varepsilon/D \le 5$ x10^{-2} and 4 x $10^{3} \le Re \le 10^{8}$	Haaland (Haaland, 1983)
(10)	$f=\left[-2\log\left(0.27(\varepsilon/D)+\frac{5.62}{Re^{0.9}}\right)\right]^{-2}$ 1 x $10^{-5} \le \varepsilon/D \le 1$ x10^{-2} and 4 x $10^{3} \le Re \le 4$ x 10^{7}	Robaina (Robaina, 1992)
(11)	$f=\left[-1.52\log\left(\left(\frac{\varepsilon/D}{7.21}\right)^{1.042}+\left(\frac{2.731}{Re}\right)^{0.9152}\right)\right]^{-2.169}$ $0 \le \varepsilon/D \le 5$ x10^{-2} and 2.1 x $10^{3} \le Re \le 10^{8}$	Ghanbari - Farshad - Rieke' (Ghanbari et al., 2011)
(12)	$f=\left[-1.8229791\log\left(\left(\frac{\varepsilon/D}{3.7315}\right)^{1.0954}+\left(\frac{5.9802}{Re}\right)^{0.9695}\right)\right]^{-2}$ $0 \le \varepsilon/D \le 9$ x10^{-2} and 3 x $10^{3} \le Re \le 9$ x 10^{8}	Beluco - Schettini (Beluco & Beatriz Camano Schettini, 2016)
(13)	$f=\left[1.805\log\left(\frac{(\varepsilon/D)^{1.108}}{4.267}+\frac{5.164}{Re^{0.966}}\right)\right]^{-2}$ $10^{-6} \le \varepsilon/D \le 0.05$ and 2 x $10^{3} \le Re \le 10^{8}$	Azizi – Homayoon - Hojjati (Azizi et al., 2019)

Take note that Jaric et al. (2011) and Fang, Xu and Zhou (2011) made a mistake in citing equation (3). They give the power 0.9 only for the Re. Fang, Xu and Zhou (2011) also made mistakes in citing equation (6) as they mistakenly mention the constant 5.74 as 5/74. Romeo, Royo and Monzón (2002) made a mistake in citing equation (7). They mistakenly mention the constant as 0.27 instead of 0.135.

2. Methodology

The data were calculated for the Reynolds number, Re, between 4,000 to 1 x 10^8. The number of nodes calculated as stated in Table 2 means the Reynolds number was calculated at 43 points as tabulated in Table 2. While for relative roughness, ε/d, the range was set between 0 to 0.10. The number of nodes calculated as stated in Table 3, which means the relative roughness was calculated at 47 points as tabulated in Table 3. The total number of nodes will be 2021 as the data then were compared with the Colebrook's equation.

As some of the equation give their suitability range of Reynolds number and relative roughness for the equation, another set of total nodes were calculated based on the range given by the equations in Table 1.

Table 2. Number of nodes for Reynolds numbers

Reynolds number, Re	**Step**	**Nodes**
4,000-10,000	1,000	7
20,000-100,000	10,000	9
200,000-1,000,000	100,000	9
2,000,000-10,000,000	1,000,000	9
20,000,000-100,000,000	10,000,000	9
	Total Nodes	43

Table 3. Number of nodes for relative roughness, ε/d

Relative roughness, ε/d	**Step**	**Nodes**
0.000000 - 0.000009	0.000001	10
0.00001 - 0.00009	0.00001	9
0.0001 - 0.0009	0.0001	9
0.001 - 0.009	0.001	9
0.01 - 0.10	0.01	10
	Total Nodes	47

All the friction factor values calculated from the equation then being compared with the Colebrook's equation. Absolute error, relative percentage error, mean absolute error (MAE), mean square error (MSE), and root mean square error (RMSE) are parameters used to compare the equation's accuracy.

Absolute error,δ_{ae} is given by

$$\delta_{ae} = \left| f_{e,i} - f_{c,i} \right| \quad (14)$$

Where, $f_{e,i}$ is the estimated friction factor calculated by the equations, while $f_{c,i}$ is the friction factor from Colebrook's equation. This equation shows the difference between the friction factor values calculated by using the equation and Colebrook's equation. The bigger the value indicates that the higher the error of the equation.

Relative percentage error,δ is given by,

$$\delta = \left| \frac{f_{e,i} - f_{c,i}}{f_{c,i}} \right| x\,100\% \tag{15}$$

This equation will be useful to see the percentage of error instead of the error itself. This parameter shows that when the relative percentage error near zero, the equation is as accurate as the Colebrook's equation. But in this paper, only the maximum value of relative percentage error, δ_{max} will be shown, which is given by

$$\delta = \delta_{\max} \tag{16}$$

Another way to see the error is by the average relative percentage error δ_{avg}. This error is good to see the overall performance of the equation. The smaller the percentage error means that the equation can fit the Colebrook's equation at most Reynolds number and relative roughness ranges. This measurement will be used as the main factor in deciding the equation's quality in this paper. Average relative percentage error is given by

$$\delta_{avg} = \frac{1}{n}\sum_{i=1}^{n}\left(\left| \frac{f_{e,i} - f_{c,i}}{f_{c,i}} \right| x\,100\% \right) \tag{17}$$

Mean absolute error (MAE) is a parameter used to measures the absolute average distance between the real data and the predicted data. When the value of MAE is more immense, it means that the equation can fit well with the actual data. In this case, the Colebrook's equation. The weakness of MAE is that it fails to punish large errors in prediction. MAE is given by equation (18).

$$MAE = \frac{1}{n}\sum_{i=1}^{n}\left| f_{e,i} - f_{c,i} \right| \tag{18}$$

To evaluate the equation by having a parameter that can punish the large error, mean square error (MSE) can be used. But the disadvantage of MSE is that it also squares up the units of data as well. So, evaluation with different units is not at all justified. MSE is given by equation (19)

$$\delta_{mse} = \frac{1}{n}\sum_{i=1}^{n}\left(f_{e,i} - f_{c,i} \right)^2 \tag{19}$$

The weakness of MSE can be improved by using the root mean squared error (RMSE). RMSE is given by equation (20), which is the square root of MSE. This metrics solves the problem of squaring the units.

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(f_{e,i} - f_{c,i} \right)^2} \tag{20}$$

The statistical analysis was used to compare the equations' accuracy by considering all ranges of Reynolds number and relative roughness ranges. It can ensure a better understanding of all the single iteration explicit approximations of Colebrook's equations. High error at a particular location should also be considered in the analysis.

3. Results and Discussion

Tables 4 and 5 show the statistical analysis results. Table 4 is the data without considering the validity range of the equations. As in Table 5, only statistical data based on the validity range of the equations.

Table 4, Altshul (2) and Round (7) equations are the worst equations giving the highest value of error regardless of the type of statistical measurements.

Surprisingly, Ghanbari–Farshad–Rieke' equation (11) developed in 2011, not giving any better result as the errors still among the highest. The same goes for Eck (4) equation. While equations (3), (5), (6), (8), and (10) give moderate errors. These equations can be used where the accuracy of the friction factor is not in priority.

Beluco-Schettini (12) is the best equation giving the lowest error but should be used with care at rough and low Reynolds number turbulent region. This equation gives the biggest error at Reynold number of 1×10^8 and relative roughness 9×10^{-2}. Azizi-Homayoon-Hojjati (13) also gives among the lowest value of the error, accept is gives a significant value of δ_{max}. Meanwhile, even though Haaland equation (9) is behind equation (12) and (13), this equation gives the lowest value of the δ_{max}. Meaning that equation (7) is the most consistent equation over the Reynolds number and relative roughness ranges.

Table 4 also can be concluded that most of the equations fail to predict correctly at the low Reynolds number turbulent region. It can be seen when most of the equation giving the highest absolute error, δ_{ae} at Reynolds number of 4000. In contrast, equation (2), (7) and (11) gives the highest absolute error at high Reynold number, 1×10^8.

The maximum value of relative percentage error, δ_{max}, at most of the time, differs from the location of highest absolute error, δ_{ae} located at. No specific relationship was found. A similar thing also happens in Table 5.

When comparing with considering the validity range as in Table 5, equation Altshul (2), Round (7), and Eck (4) equations are still the worst equations giving the highest value of error regardless of the type of statistical measurements. Ghanbari–Farshad–Rieke' equation (11) not showing significant improvement when considering the range's validity. This equation still falls under the equation category should be avoided as it's still giving a high value of error.

Not many changes for the best equation as equation (12), (13), and (9) are the equation with small error, respectively. While the best equation is (13) followed by (9) and (12) if it is seen in view of maximum absolute error,δ_{ae}.

Other equations, equation (3), (5), (6), (8), and (10), also give moderate errors, similar when validity range was not considered meaning that these equations can be used where the accuracy of friction factor is not in priority. For the maximum value of relative percentage error, δ_{max}, the value occurs mostly at the same location as where the highest absolute error, δ_{ae} located at.

An interesting point that can be seen from both of the tables is that there are no significant changes in the equation's ranking despite the validity range consideration. This means that even though the equations were developed for a specific range of Reynold number and relative roughness, it can still be used beyond the range suggested with extra precaution.

It should be noted only equation (12) was developed to cater the rough surface up to $\varepsilon/d = 9 \times 10^{-2}$.

4. Conclusion

Equations Altshul (2), Round (7) and Eck (4) should be avoided. These three equations give a huge error in the calculation. The best equation by comparing all the parameters are Beluco-Schettini (12), Azizi-Homayoon- Hojjati (13), and Haaland (9). Haaland equation (9) is the most consistent. It gives about a similar value of error regardless of the type of statistical error measurements.

Table 4. Statistical parameters for observed equations without considering the validity range

Author (year)	$\delta_{ae,max}$ **(ε / D, Re)**	δ_{max} **(ε / D, Re)**	δ_{avg}	**MAE**	**MSE**	**RMSE**
Altshul (1952)	0.1866 (9×10^{-2}, 1×10^{8})	194.0611 (9×10^{-2}, 1×10^{8})	63.6476	0.0296010403	0.0030348662	0.0550896198
Churchill (1973)	0.0027 (9×10^{-2},4000)	3.4173 (2×10^{-2},4000)	0.5418	0.0001841643	0.0000001529	0.0003909800
Eck (1973)	0.0035 (9×10^{-2},4000)	9.6240 (0, 1×10^{8})	2.1868	0.0004862699	0.0000005983	0.0007735178
Jain (1976)	0.0025 (9×10^{-2},4000)	3.1865 (2×10^{-2},4000)	0.4972	0.0001709186	0.0000001217	0.0003488373
Swamee – Jain (1976)	0.0027 (9×10^{-2},4000)	3.3536 (2×10^{-2},4000)	0.5286	0.0001787241	0.0000001452	0.0003811156
Round (1980)	0.0118 (9×10^{-2}, 1×10^{8})	12.3462 (9×10^{-2}, 1×10^{8})	3.9305	0.0015650430	0.0000099711	0.0031577002
Pavlov-Romankov-Noskov (1981)	0.0026 (9×10^{-2},4000)	3.0467 (3×10^{-2},4000)	0.4880	0.0001565994	0.0000001149	0.0003390381
Haaland (1983)	0.0009 (9×10^{-2},4000)	1.4205 (2×10^{-4}, 100000)	0.4398	0.0001182578	0.0000000265	0.0001627311
Robaina (1992)	0.0025 (9×10^{-2},4000)	3.0108 (3×10^{-2},4000)	0.4782	0.0001531830	0.0000001100	0.0003316896
Ghanbari–Farshad–Rieke' (2011)	0.0047 (9×10^{-2}, 1×10^{8})	4.8898 (9×10^{-2}, 1×10^{8})	1.0350	0.0005068262	0.0000012815	0.0011320297
Beluco-Schettini (2016)	0.0007 (9×10^{-2},4000)	3.2883 (0, 1×10^{8})	0.2989	0.0000664139	0.0000000095	0.0000976483
Azizi-Homayoon- Hojjati (2018)	0.0006 (9×10^{-2},4000)	4.9948 (0, 1×10^{8})	0.3996	0.0000672067	0.0000000104	0.0001020932

Table 5. Statistical parameters for observed equations considering the validity range

Author (year)	$\delta_{ae,max}$ **(ε / D, Re)**	δ_{max} **(ε / D, Re)**	δ_{avg}	**MAE**	**MSE**	**RMSE**
Altshul (1952)	0.1108 (5×10^{-2}, 1×10^{8})	154.8389 (5×10^{-2}, 1×10^{8})	52.7167	0.0174580756	0.0009293279	0.0304848803
Churchill (1973)	0.0024 (5×10^{-2},6000)	3.4173 (2×10^{-2},4000)	0.5512	0.0001641330	0.0000001084	0.0003292725
Eck (1973)	0.0033 (5×10^{-2},6000)	9.6240 (0, 1×10^{8})	2.3445	0.0004874730	0.0000005780	0.0007602886
Jain (1976)	0.0019 (5×10^{-2},7000)	2.8292 (2×10^{-2},5000)	0.5217	0.0001826011	0.0000001118	0.0003343224
Swamee – Jain (1976)	0.0013 (1×10^{-2},5000)	2.8279 (1×10^{-2},5000)	0.4982	0.0001228704	0.0000000485	0.0002201385
Round (1980)	0.0060 (5×10^{-2}, 1×10^{8})	10.1796 (1×10^{-5}, 1×10^{8})	3.2739	0.0008041862	0.0000019722	0.0014043404
Pavlov-Romankov-Noskov (1981)	0.0022 (5×10^{-2},4000)	3.0467 (3×10^{-2},4000)	0.4948	0.0001361540	0.0000000740	0.0002719686
Haaland (1983)	0.0006 (5×10^{-2},4000)	1.4205 (2×10^{-4}, 100000)	0.4498	0.0001057358	0.0000000208	0.0001441034
Robaina (1992)	0.0013 (1×10^{-2},4000)	2.6399 (1×10^{-2},4000)	0.4797	0.0001340140	0.0000000541	0.0002326633
Ghanbari–Farshad–Rieke’ (2011)	0.0021 (5×10^{-2}, 1×10^{8})	2.8962 (5×10^{-2}, 1×10^{8})	0.7563	0.0002213053	0.0000001736	0.0004166961
Beluco-Schettini (2016)	0.0007 (9×10^{-2},4000)	3.2883 (0, 1×10^{8})	0.2989	0.0000664139	0.0000000072	0.0000847714
Azizi-Homayoon- Hojjati (2018)	0.0004 (5×10^{-2},4000)	3.0598 (1×10^{-6}, 2×10^{7})	0.3878	0.0000619125	0.0000000083	0.0000911957

5. References

Azizi, N., Homayoon, R., & Hojjati, M. R. (2019). Predicting the colebrook-white friction factor in the pipe flow by new explicit correlations. *Journal of Fluids Engineering, Transactions of the ASME*, *141*(5), 1–8. https://doi.org/10.1115/1.4041232

Beluco, A., & Beatriz Camano Schettini, E. (2016). An Improved Expression for a Classical Type of Explicit Approximation of the Colebrook White Equation with Only One Internal Iteration. *International Journal of Hydraulic Engineering*, *5*(1), 19–23. https://doi.org/10.5923/j.ijhe.20160501.03

Churchill, S. W. (1973). Empirical expressions for the shear stress in turbulent flow in commercial pipe. *AIChE Journal*, *19*(2), 375–376. https://doi.org/10.1002/aic.690190228

Colebrook, C. F. (1939). Turbulent Flow in Pipes, With Particular Reference To the Transition Region Between the Smooth and Rough Pipe Laws. *Journal of the Institution of Civil Engineers*, *11*(4), 133–156. https://doi.org/10.1680/ijoti.1939.13150

Fang, X., Xu, Y., & Zhou, Z. (2011). New correlations of single-phase friction factor for turbulent pipe flow and evaluation of existing single-phase friction factor correlations. *Nuclear Engineering and Design*, *241*(3), 897–902. https://doi.org/10.1016/j.nucengdes.2010.12.019

Ghanbari, A., Farshad, F., & Rieke, H. H. (2011). Newly developed friction factor correlation for pipe flow and flow assurance. *Journal of Chemical Engineering and Materials Science*, *2*(6), 83–86.

Haaland, S. E. (1983). Simple and Explicit Formulas for the Friction Factor in Turbulent Pipe Flow. *Journal of Fluids Engineering*, *105*(1), 89–90. https://doi.org/10.1115/1.3240948

Jaric, M., Genić, S., Arandjelović, I., Kolendić, P., Jarić, M., Budimir, N., & Genić, V. (2011). A Review of Explicit Approximations of Colebrook's Equation. *FME Transactions*, *39*(2), 39–67.

Levenspiel, O. (1998). Engineering Flow and Heat Exchange. In *Antimicrobial Agents and Chemotherapy* (Vol. 58, Issue 12). Springer US. https://doi.org/10.1007/978-1-4899-0104-0

Nekrasov, B. B. (1969). *Hydraulics For Aeronautical Engineers*. Peace Publisher.

Robaina, A. D. (1992). Análise De Equações Explicitas Para O Cálculo Do Coeficiente "F" Da Fórmula Universal De Perda De Carga. In *Ciência Rural* (Vol. 22, Issue 2, pp. 157–159). https://doi.org/10.1590/s0103-84781992000200006

Romeo, E., Royo, C., & Monzón, A. (2002). Improved explicit equations for estimation of the friction factor in rough and smooth pipes. *Chemical Engineering Journal*, *86*(3), 369–374. https://doi.org/10.1016/S1385-8947(01)00254-6

Round, G. F. (1980). An Explicit Approximation for the Friction Factor Reynolds Number Relation for Rough And Smooth Pipes. *The Canadian Journal of Chemical Engineering*, *58*(February), 122–123.

Swamee, P. K., & Jain, A. K. (1976). Explicit Equations For Pipe-Flow Problems. *ASCE J Hydraul Div*, *102*(5), 657–664.

Winning, H. K., & Coole, T. (2013). Explicit friction factor accuracy and computational efficiency for turbulent flow in pipes. *Flow, Turbulence and Combustion*, *90*(1), 1–27. https://doi.org/10.1007/s10494-012-9419-7